Universality of the off-equilibrium response function in the kinetic Ising chain

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(Received 30 January 2002; published 20 June 2002)

The off-equilibrium response function $\chi(t,t_w)$ and autocorrelation function $C(t,t_w)$ of an Ising chain with spin-exchange dynamics are studied numerically and compared with the same quantities in the case of spin-flip dynamics. It is found that, even though these quantities are different in the two cases, the parametric plot of $\chi(t,t_w)$ versus $C(t,t_w)$ is the same. While this result could be expected in higher dimensionality, where $\chi(C)$ is related to the equilibrium state, it is far from trivial in the one-dimensional case where this relation does not hold. The origin of the universality of $\chi(C)$ is traced back to the optimization of domains position with respect to the perturbing external field. This mechanism is investigated resorting to models with a single domain moving in a random environment.

DOI: 10.1103/PhysRevE.65.066114 PACS number(s): 05.50.+q, 64.75.+g, 05.40.−a, 05.70.Ln

I. INTRODUCTION

The generalization of the fluctuation-dissipation theorem (FDT) to slowly relaxing systems, such as glasses, is an issue of foremost importance for understanding nonequilibrium processes. Ordinary FDT relates the autocorrelation $C(t,t_w)$ and the integrated response $\chi(t,t_w)$ functions, which both depend in equilibrium on the time difference $t-t_w$, where t_w is the time elapsed after the sample preparation. A general feature of slow kinetics, instead, is the aging property, namely, the dependence of the time scale of relaxation on the time t_w. This feature generally shows up both in $C(t,t_w)$ and $\chi(t,t_w)$. In the context of mean-field models for spin glasses it was shown [1] that, for large t_w, $\chi(t,t_w)$ depends on the two times through the autocorrelation function alone,

$$\chi(t,t_w)=\chi[C(t,t_w)]. \tag{1}$$

This property holds quite generally in a wide class of aging systems where deviations from the ordinary equilibrium FDT, namely, $T\chi(C)=C(t,t)-C(t,t_w)$, result in a nontrivial fluctuation-dissipation ratio $X(C)=-Td\chi(C)/dC$. Recently, a theorem has been proved [2] linking $X(C)$ to static properties,

$$\left.\frac{dX(C)}{dC}\right|_{C=q}=P(q), \tag{2}$$

where $P(q)$ is the equilibrium probability distribution of the overlaps. This opens the way to a classification of aging systems according to the structure of their equilibrium states [3] and to the recognition that the properties of the response to a perturbation are universal in systems sharing the same overlap distribution.

The ordering process of ferromagnetic systems provides a simplified framework for the study of the off-equilibrium FDT because the main features of slow relaxation are fully exhibited but the structure of their equilibrium state is simple and exactly known. While Eq. (1) is generally obeyed [3–5], the validity of Eq. (2) in this case depends on dimensionality. For $d>1$, Eq. (2) holds asymptotically, implying the same fluctuation-dissipation ratio, and hence of $\chi(C)$, for all these systems [3]. This applies, in particular, to different dynamical realizations of the same Hamiltonian model such as ferromagnets with nonconserved (NCOP) or conserved (COP) order parameter [4]. However the picture is totally different in the case of the Ising chain. With NCOP, Eq. (2) is not obeyed [5] and, instead, a nontrivial $X(C)$ is found [6] that cannot be connected to static properties.

These results suggest that the nature of $X(C)$ is essentially dynamical in this case. A natural question, then, is about which properties of the kinetics are reflected by $\chi(C)$. In order to address this point, we study in this paper the response of the one-dimensional Ising model with Kawasaki spin-exchange dynamics quenched to a low temperature, in the scaling regime preceding equilibration. In this case the order parameter is conserved and the microscopic mechanism whereby coarsening of domains is produced differs completely from the dynamics with single spin flip. With NCOP interfaces are independent Brownian walkers whose density is progressively reduced due to annihilation events. With COP, instead, the motion of interfaces is mediated by evaporation, diffusion, and recondensation of single monomers. Despite this completely different character of the dynamics, we show that $\chi(C)$ is the same for COP and NCOP. Due to the violation of the hypotheses of theorem (2) this universal character cannot be traced back to statics but it is more likely to have a common dynamical origin. Since the basic coarsening mechanisms with COP or NCOP are profoundly different, other kinetic properties, of a more general and fundamental nature, determine $\chi(C)$. The analysis carried out in this article shows that the total response can be viewed as due to the elementary contributions given by single domains. In the kinetic process domains coarsen and translate and the complex interplay between these two mechanisms produces the response. Starting from this idea we introduce simple models where a single domain is al-

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lowed to diffuse in a random environment. In this framework, the elementary response generated by the domain can be studied and from its knowledge the behavior of the original Ising chain is inferred by means of scaling arguments. The details of the rules for the motion do not change the overall behavior of the response. This approach provides a clear physical interpretation of how the response is produced in the Ising model.

These considerations are of a general nature and apply, in principle, to any dimension. On the other hand, the occurrence in $d=1$ of an off-equilibrium response that never vanishes, as opposed to the cases with $d>1$, is due to the special character of domain walls motion in one dimension. Actually, in $d>1$ the evolution of a domain is the result of two competing drives. The first is the tendency to lower surface tension by making interfaces straight. The second is the drift towards regions where the random field is favorable. For long times the first mechanism always prevails, and the response generated by the drift of domains is negligible [5]. In $d=1$, instead, domain walls are pointlike and surface tension does not play any role; moreover the drift mechanism is so efficient as to generate a nonvanishing response even in the limit of large times, when the interface density decreases to zero.

This paper is organized in six sections. Section II is devoted to a description of the COP dynamics of the unperturbed Ising chain. In Sec. III the effects of a perturbation are discussed and the response function is introduced, showing the analogy with the NCOP dynamics and the universality of $\chi(C)$. Models for a single diffusing domain are discussed in Secs. IV and V, where scaling arguments are presented to illustrate the common origin of the response within the two types of dynamics. In Sec. VI we discuss the relevance of our results for different systems and draw some conclusions.

II. UNPERTURBED DYNAMICS

We consider the one-dimensional Ising model with ferromagnetic nearest-neighbor coupling constant, whose Hamiltonian is

$$\mathcal{H}_0(\{s_i\}) = -J\sum_{i=1}^{N} s_i s_{i+1}, \tag{3}$$

where $s_i = \pm 1$. The system is quenched from an uncorrelated high temperature equilibrium state to the final temperature T. Evolution takes place through Kawasaki spin-exchange dynamics, i.e., swaps between antiparallel nearest-neighbor spins. In this way magnetization is a conserved quantity. The model describes lattice gases or binary alloys. The probability of exchanging s_i, s_{i+1} is assumed to be

$$p_T = \min[e^{-\Delta E/T}, 1], \tag{4}$$

where T is measured in units of the Boltzmann constant and $\Delta E = 2J(s_{i-1}s_i + s_{i+1}s_{i+2})$ is the energy change.

The basic features of the dynamics following an instantaneous quench are discussed in Ref. [7]. Depending on the energy change elementary moves can be distinguished into three classes with $\Delta E = 4J, 0, -4J$. The first kind of process is evaporation, namely, the separation of a spin from the boundary of a domain. Processes with $\Delta E = 0$ are the diffusion of a single spin (monomer) in the bulk of a domain of the other phase. When a diffusing monomer reaches an interface a condensation event occurs: The spin joins a domain. This process implies an energy change $\Delta E = -4J$.

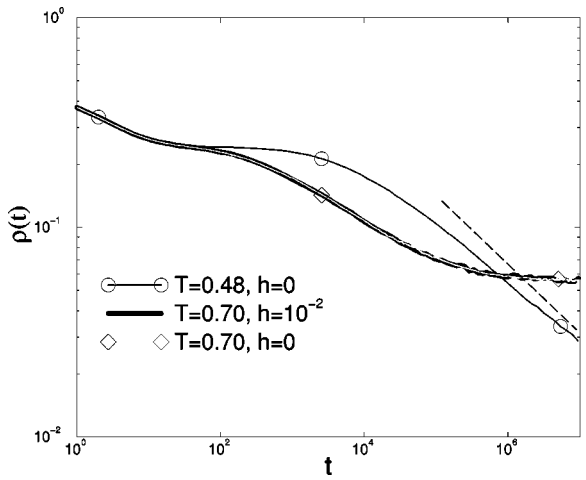

FIG. 1. The kink density ρ is plotted against time (measured in Monte Carlo steps) for quenches with $T=0.48$ and $h=0$ (black solid line) and $T=0.7$, both with the perturbation ($h=10^{-2}$, black bold solid line) and without ($h=0$, white solid line collapsing on the curve with $h=10^{-2}$). Data refer to numerical simulations of a system of $N=10^5$ spins, averaged over 10 realizations. The dashed line represents the $t^{-1/3}$ law.

Evaporation is an activated process occurring over a characteristic time $\tau_{ev} = \exp(4J/T)$. For low temperatures τ_{ev} is large and one observes a long interval $t \ll \tau_{ev}$ during which evaporation practically does not happen. In this regime a reduction of the kink density $\rho(t)$ can be obtained only by the diffusion and condensation of the monomers present in the initial state. In order to do this single spins move a distance of the order of the initial coherence length ξ in a typical time ξ^2. This leads to a decay [7,8] of the density of diffusing single spins over a characteristic time $\tau_s \sim \xi^2$. In the case we are concerned with, a quench from very high temperature, τ_s is of order unity and one observes a fast decay to a plateau on short time scales (Fig. 1). In the regime $\tau_s \ll t \ll \tau_{ev}$ no diffusing spins are left and $\rho(t)$ remains constant. At times of order τ_{ev} evaporation events begin to occur and the dynamics *restarts* (Fig. 1). Evaporated spins diffuse and they may recondense on a kink different from the one where they were emitted. This is the well known mechanism leading to the decay $\rho(t) \propto (t/\tau_{ev})^{-1/3}$ [7]: in this regime, dynamic scaling is obeyed. This behavior lasts until $t \simeq \tau_{eq}^{COP}$ such that $\rho(\tau_{eq}^{COP}) = \rho_{eq} \simeq \exp(-2J/T)$, the equilibrium kink density. At this time domains reach a size such that a second monomer is emitted when the first one is still diffusing. When they meet they form a stable dimer and this process exactly balances the domain annihilation due to the evaporation-condensation mechanism, so that ρ keeps its equilibrium value. Regarding the data presented in Fig. 1, for $T=0.7$ the exponent 1/3 is not clearly observed because the

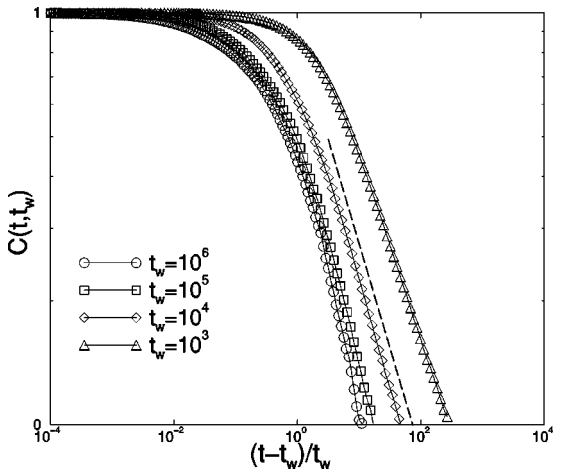

FIG. 2. The autocorrelation function $C(t,t_w)$ is plotted against rescaled time $(t-t_w)/t_w$. The dashed line represents the law $[(t-t_w)/t_w]^{-1/2}$.

system equilibrates too soon. For $T=0.48$ the effective exponent gradually decreases toward 1/3. At the longest times simulated the effective exponent is $\simeq 0.3$.

The autocorrelation function is defined as

$$C(t,t_w)=\frac{1}{N}\sum_{i=1}^{N}\langle s_i(t)s_i(t_w)\rangle, \tag{5}$$

where $\langle\cdots\rangle$ indicates thermal averaging. This quantity is shown in Fig. 2. $C(t,t_w)$ strongly depends on the range of times (t_w,t) considered. In the case $t_w \ll \tau_{eq}^{COP}$ considered here the behavior of the autocorrelation function is different for $t \ll \tau_{eq}^{COP}$ or $t \gg \tau_{eq}^{COP}$. For $t \ll \tau_{eq}^{COP}$ $C(t,t_w)$ decays as $[\rho(t)/\rho(t_w)]^{\lambda}$. The exponent λ depends on t_w as follows: When $t_w=0$, the upper bound $\lambda \leqslant d$, originally proposed by Fisher and Huse [9] provides the correct value $\lambda=1$ for the conserved $d=1$ Ising model, as shown analytically and numerically in Ref. [10]. Instead, for t_w chosen well inside the scaling regime, Yeung, Rao, and Desai [11] found a lower bound $\lambda \geqslant 3/2$ for $d=1$. Since $\lambda=1$ for $t_w=0$ this constraint implies the dependence of λ on t_w. To our knowledge, there are no results for the actual value of this exponent when $t_w \neq 0$. From the data presented in Fig. 2 one observes a power-law decay consistent with $C(t,t_w)\sim(t_w/t)^{1/2}$ for the cases $t_w=10^4,10^5$. Recalling that $\rho(t)\sim t^{-0.3}$ in the range of time considered, one obtains $\lambda \simeq 1.66 > 3/2$. In this way we show that the lower bound determined in Ref. [11] is correct and that the value of λ with t_w chosen inside the scaling regime is different from the case with $t_w=0$. Actually, the value $\lambda \simeq 1.66$ may indicate that the value $\lambda=3/2$ could be asymptotically correct. In order to check this point lower temperatures and larger waiting times should be considered. By plotting $C(t,t_w)$ against $\rho(t)/\rho(t_w)$ we have also checked that the curve with $t_w=10^3$ gives the same exponent $\lambda>3/2$, whereas the smaller exponent observed in Fig. 2 is simply due to $\rho(t)$ decaying with an effective exponent considerably smaller than 1/3 in the range of times plotted in the figure. For $t_w=10^6$ the curve starts decaying with the same exponent $\lambda=3/2$ but then the decrease becomes faster, indicating that the system is close to reaching equilibrium. A second important observation, regarding Fig. 2, is the convergence of $C(t,t_w)$ towards the scaling behavior $C(t,t_w)=\hat{C}(t_w/t)$, as expected quite generally for slow relaxation [12] and in particular for coarsening systems [13]. However, differently from NCOP, the convergence in this case is very slow and very large t_w must be considered in order to exhibit a good data collapse.

III. RESPONSE TO A PERTURBATION

Let us consider the Ising model quenched to temperature T in zero field. At time t_w a random field

$$h_i=h\epsilon_i \tag{6}$$

is applied, so that the Hamiltonian is changed into

$$\mathcal{H}=\mathcal{H}_0-\sum_{i=1}^{N}h_i s_i. \tag{7}$$

The field takes randomly only two values, $\epsilon_i=\pm 1$, with expectations

$$\overline{\epsilon_i}=0, \tag{8}$$

$$\overline{\epsilon_i\epsilon_j}=\delta_{i,j}. \tag{9}$$

The probability of exchanging two spins s_i, s_{i+1} is Eq. (4) with $\Delta E=2J(s_{i-1}s_i+s_{i+1}s_{i+2})+2h(s_i\epsilon_i+s_{i+1}\epsilon_{i+1})$. We consider h/T sufficiently small in order to be in the linear response regime. We are interested in the scaling regime, i.e., times such that $\rho(t) \gg \rho_{eq}$. Moreover we want the qualitative features of the dynamics, presented above, to be unaffected by the external field. This imposes an additional constraint $\rho(t) \gg \xi^{-1}(h)$, where $\xi(h)=4J^2/h^2$ is the Imry-Ma length [14]. For the values of h and T considered, $\xi^{-1}(h) \ll \rho_{eq}$ so that $\rho(t)$ is unchanged by the presence of the random field from the instant of the quench up to equilibration (Fig. 1).

We consider the integrated response function

$$\chi(t,t_w)=\lim_{h/T\to 0}\frac{1}{Nh}\sum_{i=1}^{N}\overline{\epsilon_i\langle s_i\rangle_h}, \tag{10}$$

where $\langle\cdots\rangle_h$ denotes average in presence of the external field. Before discussing the response of the model with COP, let us briefly recall the behavior with NCOP. In this case equilibrium is reached because spins are flipped spontaneously in the bulk of ordered domains; the characteristic time for this process is $\tau_{eq}^{NCOP}=\exp(4J/T)$. The asymptotic value of the response function is the equilibrium susceptibility

$$\chi_{eq}=1/T, \tag{11}$$

namely, $\lim_{t\to\infty}T\chi(t,t_w)=1$. In equilibrium the bulk of domains is responsible for the response because spins antialigned with the random field are more likely to be reversed by thermal excitations. However, for times $t<\tau_{eq}^{NCOP}$ spins in the bulk do not flip; the response observed in this regime is

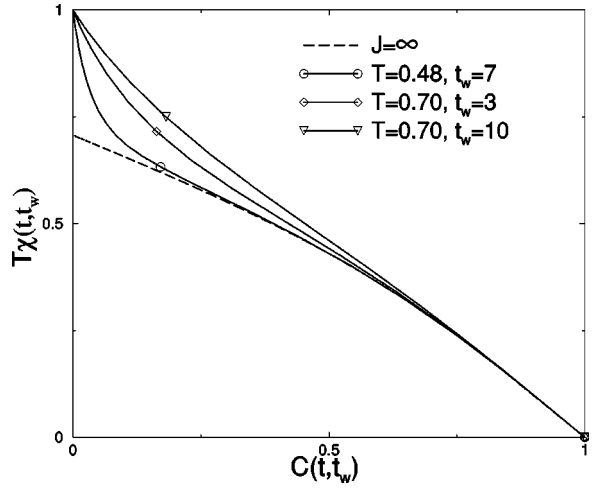

FIG. 3. The integrated response function is plotted against the autocorrelation function with NCOP. The dashed line is the case with $J=\infty$.

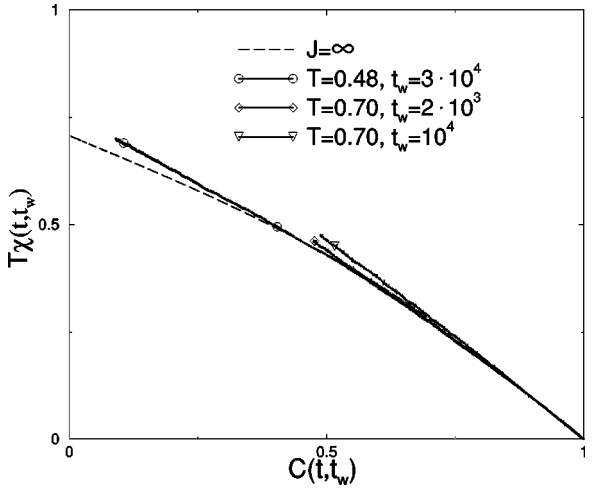

FIG. 4. The integrated response function is plotted against the autocorrelation function with COP. Data are obtained from numerical simulations of a system of $N=10^5$ spins with $h=10^{-2}$, averaged over 4000 realizations. The dashed line is the case with NCOP and $J=\infty$, as in Fig. 3.

then of nonequilibrium nature. Since the bulk is frozen, $\chi(t,t_w)$ must be provided by interfaces. Specifically the motion of kinks is such as to optimize the position of domains with respect to the random field, building up a finite response [5]. The range of times over which the nonequilibrium pattern is observed can be expanded by letting $T\to 0$ (keeping h/T small, for linear response theory to hold), or $J\to\infty$. We refer to the latter limit for simplicity, that has the advantage of being easily implemented numerically by forbidding the flip of spins in the bulk. With $J=\infty$, $\tau_{eq}^{NCOP}=\infty$ and the system never equilibrates. The exact solution of the model [6] with $J=\infty$ yields the aging form

$$T\chi(t,t_w)=\frac{\sqrt{2}}{\pi}\arctan\sqrt{\frac{t}{t_w}-1} \tag{12}$$

that converges, in the large t limit, to

$$T\chi_\infty=\frac{1}{\sqrt{2}} \tag{13}$$

for large t. For finite J, as already anticipated, the response of the model is the same as with $J=\infty$ for times $t\ll\tau_{eq}^{NCOP}$ while, for larger times, the equilibrium susceptibility is recovered.

For what concerns the generalization (1) of the FDT, notice that Eq. (12) obeys the scaling form $\chi(t,t_w)=\hat{\chi}(t_w/t)$. Hence eliminating t_w/t with $C(t,t_w)=\hat{C}(t_w/t)$ with $J=\infty$ one finds

$$\chi(C)=\frac{\sqrt{2}}{\pi}\arctan\left[\sqrt{2}\cot\left(\frac{\pi}{2}C\right)\right]. \tag{14}$$

This curve is plotted in Fig. 3. In the limit $C(t,t_w)\to 0$, namely, $t\to\infty$, the value χ_∞ is recovered, as previously discussed. For finite J the behavior of $C(t,t_w)$ and $\chi(t,t_w)$ is unchanged with respect to the case $J=\infty$ up to τ_{eq}^{NCOP}. Therefore, in the fluctuation-dissipation plot in Fig. 3 the same curve as with $J=\infty$ is followed from $C(t_w,t_w)=1$ down to $C_{eq}=C(\tau_{eq}^{NCOP},t_w)$. For $C(t,t_w)<C_{eq}$, namely, for $t>\tau_{eq}^{NCOP}$, the system goes to equilibrium, $\chi(C)$ departs from the master curve with $J=\infty$ and approaches χ_{eq}. For fixed t_w, C_{eq} grows with temperature, because τ_{eq}^{NCOP} decreases as T is increased. Alternatively, for a given temperature, C_{eq} grows by increasing t_w. In conclusion, the master curve is followed in a wider range by decreasing t_w or T, because in this way C_{eq} is reduced. This explains the behavior of $\chi(C)$ in Fig. 3.

For Kawasaki dynamics the kinetic process is more complex than in the nonconserved case. The different coarsening mechanism does not simply change the growth law exponent, but even the two-time quantities considered here are radically modified with respect to NCOP. In particular, as discussed in the preceding section, the exponent $\lambda=3/2$ differs from the value [15] $\lambda=1$ for Glauber dynamics. Given these differences the fact that the fluctuation-dissipation plot turns out to be the same for NCOP and COP, as will be shown below, is unexpected and far from being trivial.

The fluctuation-dissipation plot with COP is shown in Fig. 4. Curves with different t_w collapse on the same master curve for $C(t,t_w)>C_{eq}$. The collapse of curves with different t_w proves the validity of Eq. (1) in the scaling regime even with COP. Moreover, this master curve is the same as with NCOP. As for NCOP, the collapse occurs only when both times t and t_w belong to the scaling regime. With NCOP scaling is obeyed starting from a microscopic time t_0, which is temperature independent [16]; with COP this regime is entered after τ_{ev}, which depends on T. Therefore, for low temperatures the time scales over which curves with different t_w coincide are completely different in the two cases, as revealed from the waiting times reported in Figs. 3 and 4. Comparing the two figures, one also concludes that changing t_w or T only produces a shift of C_{eq}, the point where $\chi(C)$ deviates from the master curve, in complete analogy for both

dynamics. Then, on the basis of what is known with NCOP, we expect in the zero temperature limit the master curve to be followed down to $C(t,t_w)=0$. Let us stress that the correlation and the response functions *are different* for the two dynamics. Only when the response $\chi(t,t_w)$ is expressed as a function of the correlation $C(t,t_w)$ one obtains *the same* fluctuation-dissipation relation $\chi(C)$.

The universality of the nonequilibrium response with respect to the type of dynamics naturally raises the question of a possible common fundamental origin. As proposed in Sec. I, this must be of a dynamical character, since the connection (2) between statics and $\chi(C)$ cannot be invoked in $d=1$. The main dynamical feature of phase ordering is the presence of a coarsening structure with many competing domains. This suggests to look for the underlying universality in the response of a single domain to the perturbation. In order to test this idea, in the following sections we investigate simplified models for the motion of a single domain. Underlying this approach is the assumption that $\chi(t,t_w)$ of the whole system can be seen as the sum of the response of single domains considered independent. Correlations between domains are only responsible for the growth of their typical size. In this way we are able to identify the diffusive wandering of domains as the origin of the nonequilibrium response, both for NCOP and COP dynamics.

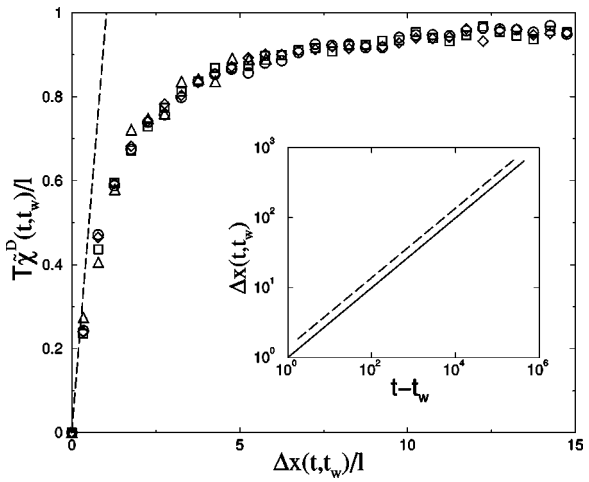

FIG. 5. The scaled single-domain response $Tl^{-1}\tilde{\chi}^{\mathcal{D}_l}(t,t_w)$ is plotted against $\Delta x(t,t_w)/l$ for $T=0.7$, $h=10^{-2}$, and $t_w=0$. Circles, squares, and diamonds correspond to domains of size $l=10,20,40$, respectively. Averages are taken over 10^6 trajectories. The dashed line is the analytic behavior for small Δx. Triangles are the response of a domain whose average size $L(t)$ grows according to $L(t)=10+\sqrt{t}/8$. In the inset the behavior of $\Delta x(t,t_w)$ is plotted for a domain of size $l=40$ with $t_w=0$. The dashed line represents the law $(t-t_w)^{1/2}$.

IV. A SINGLE-DOMAIN MODEL WITH RIGID DIFFUSION

Let us consider an isolated domain $\mathcal{D}_l$ of up spins covering the segment $[j,j+l]$ of an infinite one-dimensional lattice. A quenched random variable h_i is defined on the sites of the lattice via (6,8,9). At each time step $\mathcal{D}_l$ is allowed to move rigidly one lattice unit on the right or on the left with probability given by Eq. (4), with $\Delta E=h_j-h_{j+l+1}$ or $\Delta E=h_{j+l}-h_{j-1}$, respectively. The model can be regarded as an Ising chain with an initial condition containing l up spins in the interval $[j,j+l]$ in a sea of down spins. This Ising model is governed by a dynamical rule that conserves the magnetization and allows only rigid translations of the up domain. ΔE is then exactly the energy gain computed through the Ising Hamiltonian (7) with i running over the sites occupied by the domain.

For $h=0$ the landscape is flat and the position x of the center of the domain performs a random walk. Provided the linear response regime $(h/T\to 0)$ is considered, also for finite h the root-mean-square displacement $\Delta x(t,t_w)$ in a time interval $[t_w,t]$ obeys $\Delta x(t,t_w)\sim\sqrt{t-t_w}$, as shown in the inset of Fig. 5. The response function of the domain is defined as

$$\tilde{\chi}^{\mathcal{D}_l}(t,t_w)=\lim_{h/T\to 0}\frac{1}{h}\overline{\left\langle\sum_{i\in\mathcal{D}_l}\epsilon_i\right\rangle_h}. \tag{15}$$

The notation $\langle\cdots\rangle_h$ indicates averaging, for a single realization of the random field, over the trajectories of $\mathcal{D}_l$. Notice that, differently from Eq. (10), i runs only over sites of $\mathcal{D}_l$.

Let us consider the behavior of $\tilde{\chi}^{\mathcal{D}_l}(t,t_w)$ for short times. Initially, at $t=t_w$, the response is zero. By moving one lattice spacing the sum in Eq. (15) can change by a value $0,+2,-2$. In the first two cases the energy is unchanged or decreased; in the last one $\Delta E>0$ and the probability that such a move is accepted is $\exp(-2h/T)$. Then, averaging over the trajectories and the random field, Eq. (15) gives $T\tilde{\chi}^{\mathcal{D}_l}(t_w+1,t_w)=\lim_{h/T\to 0}[T/h]\overline{\langle\Sigma_{i\in\mathcal{D}_l}\epsilon_i\rangle_h}$ $=\lim_{h/T\to 0}T[1]-\exp(-2h/T)]/(2h)$. Taking the limit $h/T\to 0$ the linear response function follows $T\tilde{\chi}^{\mathcal{D}_l}(t_w+1,t_w)=1$.

For long times $\tilde{\chi}^{\mathcal{D}_l}(t,t_w)$ approaches, assuming equilibration, the static susceptibility

$$\tilde{\chi}^{\mathcal{D}_l}_{eq}=\lim_{h/T\to 0}\frac{T}{h}\frac{\partial\ln Z(h,T)}{\partial h}, \tag{16}$$

where

$$Z(h,T)=\sum_{k=0}^{l}p_k e^{-[h(l-2k)]/T} \tag{17}$$

is the partition function. Here k is the number of sites inside the domain where $\epsilon_i=1$ and $p_k=\binom{l}{k}2^{-l}$ is the probability of having a particular value of k. From Eq. (17) one easily finds

$$Z(h,T)=\left[\cosh\left(\frac{h}{T}\right)\right]^l \tag{18}$$

yielding $T\tilde{\chi}^{\mathcal{D}_l}_{eq}=\lim_{h/T\to 0}(lT/h)\tanh(h/T)[\cosh(h/T)]^{l-1}=l$.

The behavior of $\tilde{\chi}^{\mathcal{D}_l}(t,t_w)$, obtained numerically, is plotted in Fig. 5, showing the validity of the scaling form

$$T\tilde{\chi}^{\mathcal{D}_l}(t,t_w) = lg(y), \tag{19}$$

where $y(t,t_w) = \Delta x(t,t_w)/l$. The scaling function behaves as

$$g(y) = \begin{cases} y & \text{for } y \ll 1, \\ 1 & \text{for } y \gg 1, \end{cases} \tag{20}$$

in agreement with the analytical results for short and long times.

So far we have studied a model where the size l of the domain is conserved by the dynamical rule. However, in order to apply this result to the description of the Ising chain, where domains coarsen, we consider now a slightly modified version where the size of the single domain varies stochastically while growing on average. In such a situation, we indicate with $l(t)$ the size of the domain at time t, with $L(t)$ the average of $l(t)$, and define $y(t,t_w)$ via

$$y(t,t_w) = \Delta x(t,t_w)/L(t). \tag{21}$$

We have performed numerical simulations where the size $l(t)$ of $\mathcal{D}_l$ was increased with a stochastic rule such that $L(t) = L(0) + a\sqrt{t}$, with $L(0) = 10$ and $a = 1/8$ (these values are chosen for numerical convenience). Figure 5 shows that even in this case the scaling form (19),(20) holds.

Connection with the Ising model

From the knowledge of the response of a single domain, we can recover the behavior of the Ising chain, where many domains compete, by assuming that the latter can be adequately described by a collection of quasi-independent domains of average size $L(t)$. This means that all effects produced by correlations between domains, apart from the increase of $L(t)$, are supposed not to be relevant. This assumption will be further discussed in Sec. V.

The overall response of the Ising model is then given by

$$\chi(t,t_w) = \sum_l \mathcal{P}(l,t)\tilde{\chi}^{\mathcal{D}_l}(t,t_w), \tag{22}$$

where $\mathcal{P}(l,t)$ is the fraction of spins belonging to domains of size l at time t, which obeys [17] the scaling form $\mathcal{P}(l,t) = L(t)^{-1} f[l/L(t)]$. The analysis can be carried out more easily with the approximation $f(x) \simeq \delta(x-1)$, i.e., assuming that all domains have exactly the same size $L(t)$. Then

$$\chi(t,t_w) \simeq L(t)^{-1}\tilde{\chi}^{\mathcal{D}_L}(t,t_w) \simeq \frac{1}{T}g(y), \tag{23}$$

where $y(t,t_w)$ is defined by Eq. (21) and $\Delta x(t,t_w)$ is the average distance traveled by domains of size $L(t)$. On the basis of the large time behavior of $y(t,t_w)$, three situations can be distinguished, namely, $\lim_{t\to\infty} y(t,t_w) = \infty$, $\lim_{t\to\infty} y(t,t_w) = \text{const} > 0$ or $\lim_{t\to\infty} y(t,t_w) = 0$, giving rise to different values for the response.

The first case occurs in the equilibrium state of the Ising model because domains diffuse while their size stays constant due to the formation of new kinks. On the basis of Eq. (23), for the total response of the Ising chain one obtains $\lim_{t\to\infty} \chi(t,t_w) = \chi_{eq} = 1/T$. This is indeed the value (11) found for the original Ising chain.

The second case, $\lim_{t\to\infty} y(t,t_w) = \text{const} > 0$, occurs when $\Delta x(t,t_w)$ and $L(t)$ grow with the same exponent. This happens in the scaling regime, both with spin-flip or Kawasaki dynamics. With NCOP this is true because interfaces are Brownian walkers,

$$\Delta x(t,t_w) \sim (t-t_w)^{1/2}, \tag{24}$$

while it is well known [13] that

$$L(t) \sim t^{1/2}. \tag{25}$$

With spin-exchange dynamics the behavior of $\Delta x(t,t_w)$ can be obtained with the following argument: The displacement of domains is mediated by the evaporation and condensation of monomers. When a monomer travels a distance $L(t)$, leaving one domain and joining the nearest, one of the boundaries of each of the two domains is displaced by one lattice unit. The number of monomers per unit time that leave a domain and reach the neighbor instead of recondensing on the original one is $\rho(t)$ [7]. Hence, on average, it takes a time $\sim L(t)$ to move a domain by a unitary distance. Since domains move randomly the law of Brownian motion

$$[\Delta x(t,t_w)]^2 = 2D(t-t_w) \tag{26}$$

is obeyed, with a diffusion coefficient

$$D \sim \rho(t). \tag{27}$$

Using the appropriate growth law for COP,

$$L(t) = \rho(t)^{-1} \sim t^{1/3}, \tag{28}$$

one gets

$$\Delta x(t,t_w) \sim [(t-t_w)/t^{1/3}]^{1/2}. \tag{29}$$

Hence for long times $\Delta x(t,t_w) \sim t^{\beta}$, with $\beta = 1/2$ for NCOP and $\beta = 1/3$ for COP. Comparing Eqs. (24), (25), (28), and (29) one concludes, as already anticipated, that $\Delta x(t,t_w) \sim L(t)$ regardless of the dynamical rule. Hence, recalling Eq. (21) one obtains with *both* dynamics the universal form

$$y = y_\infty \left(1 - \frac{t_w}{t}\right)^{1/2}, \tag{30}$$

where y_∞ is a constant that may depend on the dynamical rule. Inserting this expression into Eq. (23) one recognizes that the global response obeys the scaling behavior $\chi(t,t_w) = \hat{\chi}(t_w/t)$ which, as discussed in Sec. III, is correct for the Ising chain. Moreover one also obtains $\lim_{t\to\infty} y = y_\infty < \infty$ and, therefore, an asymptotic nonequilibrium value $\chi_\infty = g(y_\infty)/T \neq \chi_{eq}$ is generated. This limiting value is different from the static susceptibility χ_{eq} because y_∞ is finite, as opposite to the equilibrium case discussed above where $\lim_{t\to\infty} y = \infty$. This is in agreement with the behavior of the original Ising model (13), with both dynamics.

Our approach does not allow the evaluation of y_∞. However it reproduces the main features of the nonequilibrium

response and offers an insight into what goes on after a perturbation has been switched on in the one-dimensional Ising model. In particular, the model points out clearly which is the mechanism whereby the response is produced. Actually, the domain responds to the perturbation by moving so as to optimize its position with respect to the random field. Such a sharp statement is made possible in this context by the fact that the dynamical rules do not allow any other possibility and indicates that the same mechanism is at work also in the Ising chain. The discussion presented insofar shows also that another physical ingredient plays a fundamental role in $d=1$: The convergence to a finite value of $y(t,t_w)$, namely, $\lim_{t\to\infty} y(t,t_w)=y_\infty\neq 0$. This property holds because the displacement of domains in $d=1$ is proportional to their average size. With these ingredients the one-domain model predicts a finite limiting value $\lim_{t\to\infty}\chi(t,t_w)=\chi_\infty$ of the response of the Ising chain. We emphasize that the property $\lim_{t\to\infty} y(t,t_w)=y_\infty\neq 0$ is far from being trivial. Although we restrict the analysis in this paper to one dimension, we believe this to hold only in the $d=1$ case. As discussed in Sec. I, in one-dimension domains diffuse in order to lower the magnetic energy in absence of the additional force produced by surface tension, because kinks are pointlike objects. In $d>1$, instead, the displacement of an interface is not only ruled by the random field but is also governed by curvature. This additional mechanism lowers surface tension and competes with the tendency to lower the magnetic energy. The weakening of the drift of domains limits their motion so that $\Delta x(t,t_w)$ grows more slowly than $L(t)$. As a result $\lim_{t\to\infty} y(t,t_w)=0$ and the response produced in this way, from Eqs. (19) and (20), vanishes. Therefore, we expect the third possible behavior of $y(t,t_w)$ introduced above to be realized for $d>1$.

V. A SINGLE-DOMAIN MODEL WITH SPIN EXCHANGE DYNAMICS

In the preceding section we have discussed the diffusion of a rod of average size $L(t)$. With this dynamics we have obtained the formula (19) for the response. The connection with the Ising model was then made possible by scaling arguments where $\Delta x(t,t_w)$ and $L(t)$ were assumed *a priori* to behave as in the NCOP or COP Ising model. The dynamics of the rod, however, is quite different from the actual behavior of the Ising model, where generally domains do not move as a whole. Furthermore, while diffusion is the mechanism whereby interfaces move in the Ising model with NCOP, as stated by Eq. (24), with COP the law (29) is obeyed, showing a non-Brownian character. Despite these shortcomings of the model, we have obtained a good description of the Ising chain and this suggests that the actual details of the dynamics are irrelevant.

In this section we introduce an improved single-domain model that takes into account the different kinetics for NCOP and COP. In particular, for COP, the evaporation-condensation mechanism that rules the evolution is taken into account.

We study an Ising chain of size N with periodic boundary conditions and two interfaces initially located in $x_1=1, x_2=N/2+1$. We consider both spin-flip and spin-exchange dynamics, and require the initial structure with only two domains of opposite sign to be preserved at all times. This condition is guaranteed if the temperature is low enough. However simulations at very low T would be numerically too demanding, in particular for COP. For this reason we implement the aforementioned condition in a different way: With NCOP we let $J=\infty$ by forbidding flips in the bulk. For COP we use a modified dynamics where new dimers are not allowed to form. Specifically, when two monomers meet, one of the two particles is removed and attached randomly to another kink. With these rules a configuration with only two domains persists.

Let us focus on one of the two domains, indicated with $\mathcal{D}$, with the center located in $(x_1+x_2)/2$. With COP the system evolves via exchanges of monomers between the two boundaries of $\mathcal{D}$. Since at low T at most one monomer is present the size $l=x_2-x_1$ of $\mathcal{D}$ is practically conserved. With NCOP, on the other hand, interfaces diffuse independently and the size changes. We consider the range of times $t\ll(N/2)^2$ so that annihilation events do not happen. Although the size of domains changes, the average value $L=\overline{\langle x_2-x_1\rangle_h}$ is constant, due to symmetry. Then $L=\text{const}$ with both types of dynamics. Our goal is to compute

$$\chi^{\mathcal{D}}(t,t_w)=\lim_{h/T\to 0}\frac{1}{Nh}\sum_{i=1}^{N}\overline{\epsilon_i\langle s_i\rangle_h}. \tag{31}$$

Notice that this is exactly the response (10) defined for the Ising model. The subscript $\mathcal{D}$ simply reminds that we are in a situation with only two domains of fixed size.

Before discussing the behavior of the model let us comment on the relationship between this approach and the one presented in the preceding section. An obvious difference is the presence of two domains instead of one. However, with the choice $L=N/2$ the domains are equivalent and this merely doubles the response. On the other hand, in the regime $t\ll(N/2)^2$ considered now, $y(t,t_w)=\Delta x(t,t_w)/L\ll 1$. Therefore, if this model is equivalent to the previous one we expect to recover the results with $y(t,t_w)\ll 1$ of the preceding section.

With NCOP the behavior of the model can be deduced from the knowledge of the response $\chi_{sing}(t,t_w)$ of the case with a single kink located in $x(t)$ and fixed boundary conditions discussed in Ref. [5]. For this system it was shown exactly that

$$\chi_{sing}(t,t_w)=\lim_{h/T\to 0}\frac{1}{Nh}\sum_{i=1}^{N}\overline{\langle s_i\rangle_h\epsilon_i}=\frac{2}{NT}\delta x(t,t_w)$$
$$\sim(t-t_w)^{1/2}, \tag{32}$$

where $\delta x(t,t_w)=\langle|x(t)-x(t_w)|\rangle$ is the average distance traveled by the kink in the time interval $[t_w,t]$. Result (32) allows one to deduce the behavior of the present model. Denoting by $\chi^{(1)}_{sing}(t,t_w)$ and $\chi^{(2)}_{sing}(t,t_w)$ the responses associated to the two interfaces the total response is simply given by

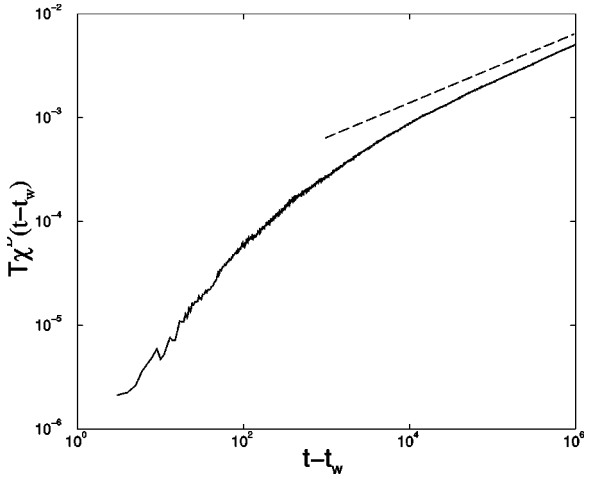

FIG. 6. The response of a pair of domains diffusing via Kawasaki dynamics with probability (37). Simulations are presented for a system of $N=10^5$ spins with $T=0.7$ and $h=10^{-2}$ averaged over 2×10^5 realizations. The dashed line is the $t^{1/3}$ behavior. Time is measured in Monte Carlo steps.

$$\chi^{\mathcal{D}}(t,t_w)\sim\chi^{(1)}_{sing}(t,t_w)+\chi^{(2)}_{sing}(t,t_w)\sim\delta x_1(t,t_w)+\delta x_2(t,t_w) \tag{33}$$

because the two interfaces are independent. Indicating with $\Delta x(t,t_w)=[\,\delta x_1(t,t_w)+\delta x_2(t,t_w)\,]/2\propto(t-t_w)^{1/2}$ the average distance traveled by $\mathcal{D}$ in the interval $[t_w,t]$ one finds

$$\chi^{\mathcal{D}}(t,t_w)\sim\Delta x(t,t_w)\sim(t-t_w)^{1/2}. \tag{34}$$

Going back to the Ising model, assuming again that domains are noninteracting, the response is obtained by multiplying $\chi^{\mathcal{D}}(t,t_w)$ times the number of domains present $\rho(t)\sim t^{-1/2}$,

$$\chi(t,t_w)\simeq\rho(t)\chi^{\mathcal{D}}(t,t_w), \tag{35}$$

yielding

$$\chi(t,t_w)=\chi_\infty\left(1-\frac{t_w}{t}\right)^{\alpha}, \tag{36}$$

where $\alpha=1/2$ and χ_∞ is the asymptotic value. Then, from Eqs. (19), (20), and (30) one recovers the behavior of the previous model in the small $y(t,t_w)$ limit, as expected.

With NCOP this result has been obtained by letting the single interface move as in the original Ising model, namely, with the same update rules for the spins. With COP this issue is more subtle. In the Ising model the diffusivity of domains depends on time via Eq. (27). In the present case, with a fixed size of the domains, D is constant. To keep this into account, we consider a spin-exchange dynamics generated by the modified probability

$$p_T(s_i,s_{i+1})=\min[n^{-1/2}(t)e^{-\Delta E/T},1], \tag{37}$$

where $n(t)$ is a counter of evaporation events. With this rule the diffusivity is proportional to $n^{-1/2}(t)\sim\rho(t)$, as in the Ising model. The response of the model, obtained by numerical simulations, is plotted in Fig. 6, and for long times

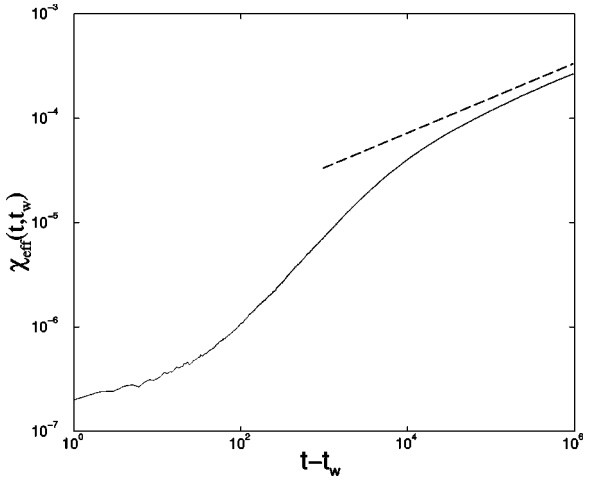

FIG. 7. The effective response of the Ising model with Kawasaki dynamics for a quench to $T=0.48$ with $t_w=3\times10^4$ and $h=10^{-2}$. Data are averaged over 4000 realizations. The dashed line is the $t^{1/3}$ behavior. Time is measured in Monte Carlo steps.

$$\chi^{\mathcal{D}}(t,t_w)\sim(t-t_w)^{1/3}. \tag{38}$$

From Eq. (38) the response of the Ising model is obtained through Eq. (35), leading to the same form (36) as for NCOP, but with $\alpha=1/3$. This shows that the present model, evolving with two different dynamical rules with NCOP or COP, gives rise to different responses. This agrees with the behavior of the Ising model, where both $\chi(t,t_w)$ and $C(t,t_w)$ are different but $\chi(C)$ is the same.

The approach in terms of single domains is based on the assumption (35) of their quasi-independence. We have shown that this hypothesis allows a description of the Ising kinetics in terms of scaling arguments and provides a good agreement with the original model. We are now in a position to substantiate further the validity of this assumption by checking the accuracy of Eq. (35).

From the knowledge of $\chi(t,t_w)$ for the Ising model we extract the effective response due to a single domain defined by

$$\chi(t,t_w)=\rho(t)\chi_{eff}(t,t_w). \tag{39}$$

The accuracy of the independent domain approximation can be determined by comparing $\chi^{\mathcal{D}}(t,t_w)$ with $\chi_{eff}(t,t_w)$. With NCOP this issue has been considered in Ref. [5] showing a very good agreement. In particular $\chi^{\mathcal{D}}(t,t_w)$ and $\chi_{eff}(t,t_w)$ both increase as $t^{1/2}$ for large t. For COP the behavior of $\chi_{eff}(t,t_w)$ is shown in Fig. 7. For large t, in particular, $\chi^{\mathcal{D}}(t,t_w)$ and $\chi_{eff}(t,t_w)$ grow with the same exponent 1/3.

The results of this section point out the robustness of the mechanism generating the response in $d=1$ which only relies on the coarsening domain structure of the system. Provided this character is maintained the global behavior of the susceptibility and, in particular, the convergence to a finite nonequilibrium value, is the same.

VI. DISCUSSION

In this paper we have studied the off-equilibrium response of the $1d$ Ising model. We have shown that the fluctuation-dissipation plot is the same with NCOP or COP. In $d=1$, where the connection (2) with statics cannot be invoked, this universal character has a dynamical origin, as shown by the analysis of simplified models presented in Secs. IV and V.

An important issue is the relevance of the picture provided by the Ising chain for arbitrary dimension. In $d=1$ with NCOP the Ising model reaches equilibration on the characteristic time $\tau_{eq}^{NCOP}=\rho_{eq}^{-2}=\exp(4J/T)$. As mentioned in Sec. III this is the time necessary for flipping spins in the bulk of domains; the same mechanism is also responsible for the equilibrium response $\chi_{eq}(t-t_w)$. The off-equilibrium response, instead, develops in the regime $t<\tau_{eq}^{NCOP}$. Hence in $d=1$ two kinds of response exist, which are observed on different time scales separated by τ_{eq}^{NCOP}. This feature gives rise to the pattern presented in Fig. 3.

The case $d>1$ presents some differences. For quenches below T_c global equilibration is never reached in an infinite system. Despite this fact the response can still be split into an equilibrium and an aging part: The bulk of domains, which behaves as a pure phase and attains local equilibrium, produces $\chi_{eq}(t-t_w)$ which obeys Eqs. (1) and (2). Domain walls, instead, are responsible for the nonequilibrium part that obeys [5] the scaling form

$$T\chi(t,t_w)=t_w^{-a}f\left(\frac{t}{t_w}\right), \tag{40}$$

with

$$a=\begin{cases}(d-1)/4 & \text{for } d<3,\\ \frac{1}{2} & \text{for } d>3,\end{cases} \tag{41}$$

and logarithmic corrections in $d=3$. The dependence of a on dimensionality results from the competition between the drift of interfaces produced by the perturbation and the force caused by their curvature. When $a=1/2$, as for $d>3$, the response is simply proportional to $\rho(t)$ implying that a single interface produces a response that does not depend on time. This is what happens if interfacial spins simply *polarize* according to the random field on a microscopic time scale. On the other hand, from the knowledge of the behavior of the one-dimensional case, we know that the wandering of interfaces gives rise to a single-interface response growing as $(t-t_w)^{1/2}$. Therefore, a natural interpretation is the following: Curvature, which is absent in $d=1$ becomes progressively more important as d increases, due to the coordination number. The attempt to lower surface tension weakens the drift of domain walls and inhibits the response mechanism associated with it. This progressively increases the value of a with respect to $d=1$ as dimensionality is increased. Then, for $d>3$ the motion of domain walls is fully governed by curvature whereas, for $d<3$ the drift mechanism partly compensate the decrease of $\rho(t)$, resulting in a smaller exponent a. Only at the lower critical dimension $d=1$, however, this mechanism is so efficient as to balance exactly the loss of interfaces yielding $a=0$ and an asymptotic finite limit χ_∞.

For $d>1$ the presence of an equilibrium and an off-equilibrium response, and the mechanisms whereby they are produced, strongly resemble the situation in $d=1$. However, while for $d=1$ they are observed on different time scales, for $d>1$ they are both developed during the phase-ordering process. Then, for $d>1$, since the equilibrium part alone obeys Eq. (2), in order for the total response to fit into the scheme (2), the off-equilibrium contribution must vanish in the large t_w limit. Equation (41) shows that this happens for $d>1$ but the decay of the off-equilibrium response for $d<3$ is slower than usually expected on the basis of the idea that the random field simply *polarizes* the interfacial spins if $d\leqslant 3$.

For $d=1$ in the phase-separation regime $\chi_{eq}(t-t_w)$ is absent and only the off-equilibrium response is developed, which in this case does not vanish for $t_w\to\infty$ and causes the breakdown of the connection (2) with the statics.

In conclusion, with NCOP an overall discussion of the response of the Ising model to stochastic perturbations can be given in terms of two mechanisms whose interplay is regulated by dimensionality. The equilibrium response, which prevails in $d>1$, only relies on the structure of the equilibrium state through Eq. (2) and, therefore, is independent from dynamics. In this paper it was shown that in $d=1$ also the off-equilibrium response is independent on the kinetic rules, although this property has a different origin. The possible universality of the out of equilibrium response in higher dimensionality and the generality of the scaling form (40) are interesting issues that deserve to be investigated in the future.

As a final comment, we discuss the possible relevance of our studies for systems with a vector order parameter with N components. In this case spins basically rotate rather than flip and this is totally different with respect to scalar systems. The absence of bulk and interfaces prevents a straightforward extension of the concepts developed in this paper and the mechanisms by which the response is built up in the vectorial case is complex and still not well understood. However, the exact computation of the response function in the solvable large-N model [18] has recently shown a pattern that resembles the behavior of scalar systems. Actually in the large-N model the response function can be explicitly split into an equilibrium and an off-equilibrium part. It can be shown, moreover, that the former satisfies Eq. (2) while the latter obeys Eq. (40) with an exponent a that vanishes at the lower critical dimension of the model which, due to the vectorial character, is $d=2$. This close analogy with the scalar case shows that, although the microscopic dynamics is different, the same competition between two mechanisms for the development of the response exists probably for every value of N and that the same scaling relation (40) holds. This suggests the idea that the off-equilibrium response may be independent from the dynamics also in vectorial systems, as we have shown for the Ising model at the lower critical dimension.

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